International Journal of Theoretical Physics, Vol. 33, No. 12, 1994

Extensions of Real-Valued Difference Posets

Radko Mesiar[1] **and Endre Pap**[2]

Received March 3, 1994

The difference poset of real-valued functions (fuzzy sets) is investigated. The difference posets on some subsets of the unit real interval [0, 1] are characterized. It is shown that although it is not always possible to represent the difference by some generator, for dense subsets such a representation exists and is unique.

1. INTRODUCTION

Very recently an abstract model of quantum mechanics was introduced by Kopka and Chovanec (n.d.; Kopka, 1992) under the name of difference poset (briefly, D-poset). This structure is a generalization of quantum logic (Kalmbach, 1983; Ptak and Pulmannová, 1991), MV-algebras (Chang, 1958; Mundici, 1986), orthoalgebras (Foulis *et al.*, 1992), the set of all effects (Dvurečenskij, n.d.), and the collection of fuzzy sets (Kopka, 1992). Although the notion of the difference poset is so general, many important results were proved on this structure (de Lucia and Pap, n.d.; Dvurečenskij, n.d.; Dvurečenskij and Pulmannová, n.d.-*a,b*; Dvurečenskij and Riečan, n.d.; Navara and Ptak, n.d.; Riečanova and Bršel, 1994).

The collection of fuzzy subsets of a given set X endowed by fuzzy connectives (Weber, 1983) induced by complementation c, a triangular norm T (Ling, 1965; Schweizer and Sklar, 1983) and a c-dual triangular conorm S forms an MV-algebra if and only if S is generated by the same normed generator g as c. By Belluce (1986), the class of fuzzy MV-algebras coincides, up to an isomorphism, with the class of Archimedean MV-algebras (semisimple MV-algebras).

The investigations of difference posets on the space of real-valued functions (e.g., fuzzy sets) leads to the study of D-posets on the real line.

[1]Slovak Technical University, 81368 Bratislava, Slovakia.
[2]Institute of Mathematics, University of Novi Sad, 21000 Novi Sad, Yugoslavia.

0020-7748/94/1200-2317$07.00/0

This general problem can be transformed by a simple transformation to the problem of D-posets on subsets of [0, 1], where the minimal element is 0 and we work then with the usual order of reals. If the difference poset consists of the whole interval [0, 1], a full characterization of the difference $\ominus$ can be found in Mesiar (n.d.-*a*,*b*). However, the underlying space need not consist of all points from [0, 1]. The aim of this paper is to study such difference posets from the extension point of view, i.e., to find some conditions ensuring the representation of the difference $\ominus$ by means of generators.

2. DIFFERENCE POSET

Definition 1. A D-poset (difference poset) is a partially ordered set $\mathbf{L}$ with a partial ordering $\leq$, maximal element $\mathbf{1}$, and a partial binary operation $\ominus: \mathbf{L} \times \mathbf{L} \to \mathbf{L}$, called a difference, such that, for $a, b \in \mathbf{L}$, $b \ominus a$ is defined if and only if $a \leq b$, for which the following axioms hold for $a, b, c \in \mathbf{L}$:

(DP_1) $b \ominus a \leq b$.
(DP_2) $b \ominus (b \ominus a) = a$
(DP_3) $a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Then there exists also a minimal element $\mathbf{0}$ $(= \mathbf{1} \ominus \mathbf{1})$.

The following properties of the operation $\ominus$ have been proved in Kopka and Chovanec (n.d.):

(a) $a \ominus \mathbf{0} = a$.
(b) $a \ominus a = \mathbf{0}$.
(c) $a \leq b \Rightarrow b \ominus a = \mathbf{0} \Leftrightarrow b = a$.
(d) $a \leq b \Rightarrow b \ominus a = b \Leftrightarrow a = \mathbf{0}$.
(e) $a \leq b \leq c \Rightarrow b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.
(f) $b \leq c$, $a \leq c \ominus b \Rightarrow b \leq c \ominus a$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.
(g) $a \leq b \leq c \Rightarrow a \leq (c \ominus (b \ominus a)) \ominus a = c \ominus b$.

3. DIFFERENCE POSET ON THE INTERVAL [0, 1]

Theorem 1. Let $([0, 1], \ominus, \leq, 0, 1)$ be a D-poset. Then and only then there is a unique generator g where g is an increasing bijection of [0, 1] onto [0, 1], such that for all a and b, $0 \leq a \leq b \leq 1$, we have

$$b \ominus a = g^{-1}(g(b) - g(a))$$

Sketch of the proof [for details see Mesiar, (n.d.-*a*)]. We extend the difference $\ominus$ for all $a, b \in [0, 1]$ putting $b \ominus a = 0$ whenever $a > b$. We show that the unary operation $\perp$ defined via

$$a^{\perp} = 1 \ominus a$$

is a strong negation on [0, 1]. Further, a binary operation $\oplus$ on [0, 1] defined via

$$a \oplus b = (b^{\perp} \ominus a)^{\perp}$$

is a t-conorm (i.e., associative, commutative, nondecreasing binary operation on [0, 1] with the natural element 0).

In the next step, we show the continuity of the extended difference $\ominus$ and consequently the continuity of the t-conorm $\oplus$. It is easy to see that then $\oplus$ is a nilpotent t-conorm (i.e., $a \oplus a > a$ for all $a \in (0, 1)$ and $a \oplus a = 1$ for some $a \in (0, 1)$). Due to Ling's representation theorem (Ling, 1965), there is a unique normed generator g such that

$$a \oplus b = g^{-1}(\min(1, g(a) + g(b))$$

Taking into account that

$$b \ominus a = (a \oplus b^{\perp})^{\perp}$$

and that $a^{\perp} = \sup(b\colon a \oplus b \leq 1)$, one gets immediately $a^{\perp} = g^{-1}(1 - g(a))$ and

$$b \ominus a = g^{-1}(\max(0, g(b) - g(a))$$

Hence if $a \leq b$,

$$b \ominus a = g^{-1}(g(b) - g(a)) \quad \text{QED}$$

Note that the proof of Mesiar (n.d.-*a*) (up to the uniqueness) can be applied to any difference poset $(M, \oplus, \leq, 0, 1)$, $M \subset [0, 1]$, $\{0, 1\} \subset M$, where M is a closed subset of [0, 1]. However, there are some D-posets on subsets of [0, 1] where the difference cannot be generated by some generator g.

Example 1. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be a strictly increasing and a strictly decreasing sequence, respectively, such that $a_0 = 0$, $b_0 = 1$, and

$$\lim_{n \to \infty} a_n = a = \lim_{n \to \infty} b_n$$

Put $M = \{a_n\}_{n=0}^{\infty} \cup \{b_n\}_{n=0}^{\infty}$ and define a difference $\ominus$ on M as follows:

$$a_n \ominus a_k = a_{n-k} \qquad \text{whenever} \quad k \leq n$$

$$b_m \ominus b_k = a_{k-m} \qquad \text{whenever} \quad m \leq k$$

$$b_m \ominus a_n = b_{m+n} \qquad \text{for all} \quad m, n = 0, 1, \ldots$$

Then $(M, \ominus, \leq, 0, 1)$ is a difference poset. Suppose that there is a generator g such that

$$b \ominus a = g^{-1}(g(b) - g(a)) \qquad \text{for all} \quad a, b \in M, \quad a \leq b$$

Note that $g(x) < 1$ for all $x < 1$ and $g(x) > 0$ for all $x > 0$. Put $g(a_1) = u > 0$. Then

$$a_1 = a_2 \ominus a_1 = g^{-1}(g(a_2) - u)$$

implies $g(a_2) = 2u$. By induction, $g(a_n) = nu$, $n = 1, 2, \ldots,$ and consequently g is not bounded on $M \subset [0, 1]$, which is a contradiction.

We have already mentioned that the closedness of M is a sufficient condition for the existence of a generator (but not for its uniqueness!) of the difference $\ominus$ on M and consequently for the extension of $\ominus$ to the whole interval [0, 1]. Of course, this condition is not necessary.

Another sufficient (but not necessary) condition is given by the following theorem. In this case the generator is unique.

Theorem 2. Let $(M, \ominus, \leq, 0, 1)$ be a difference poset and let M be a dense subset of [0, 1]. Then there is a unique generator g of the difference $\ominus$ on M, i.e., there is a unique extension of the difference $\ominus$ to the interval [0, 1].

Proof. If there is an extension of the operation $\ominus$ to [0, 1], then, by Theorem 1, this extension is generated by a unique generator g. The continuity of g and denseness of M imply the unicity of the representation of the operation $\ominus$ on [0, 1] by g. Hence it is enough to show that the extension of $\ominus$ on M to [0, 1] is possible.

We define this extension in the following way:

if $x = y \in [0, 1]$, then $y \ominus x = 0$
if $x = 0$, $y \in [0, 1]$, then $y \ominus x = y$
if $0 < x \leq y \leq 1$, then $y \ominus x = \sup(r \ominus s;\, r, s \in M, r \leq y, s \geq x)$

We have to show that this extension is really a difference.

Let us show (DP_2), i.e., $y \ominus (y \ominus x) = x$ whenever $x \leq y$. Let us denote $u := y \ominus x$. Then we have

$$y \ominus u = \sup(i_n \ominus j_m;\, i_n \leq y, j_m \geq u)$$

and

$$y \ominus x = \sup(i_n \ominus k_m;\, i_n \leq y, k_m \geq x)$$

Then we have

$$j_m \geq y \ominus x \geq i_n \ominus k_m$$

and

$$i_n \ominus j_m \le i_n \ominus (i_n \ominus k_m) = k_m$$

Then

$$y \ominus u = \sup(i_n \ominus j_m ; i_n \le y, j_m \ge u) \le k_m$$

Hence

$$y \ominus u = y \ominus (y \ominus x) \le \inf k_m = x$$

If $y \ominus (y \ominus x) < x$, then we take $a \in M$ such that

$$y \ominus (y \ominus x) < a < x$$

On the other hand, we have by $a < x \le k_m$ the inequality

$$y \ominus x = \sup(i_n \ominus k_m) \le \sup(i_n \ominus a) = y \ominus a$$

and so

$$x = y \ominus (y \ominus x) = \sup(i_n \ominus j_m) \le \sup(i_n \ominus (i_n \ominus a)) = a$$

Contradiction.

The other axioms (DP_1) and (DP_3) can be shown in a similar way.

Note that the reverse problem is also not trivial, i.e., given a D-poset $([0, 1], \ominus, \le, 0, 1)$ and a dense subset $M \subset [0, 1]$, $\{0, 1\} \subset M$, when the operation $\ominus$ can be restricted to M, i.e., when $(M, \ominus, \le, 0, 1)$ is a D-poset. The problem is whether M is closed with respect to $\ominus$. Take, e.g., $g(x) = x^2$, i.e.,

$$b \ominus a = (b^2 - a^2)^{1/2} \qquad \text{for} \quad a \le b$$

and $M = Q \cap [0, 1]$. The operation $\ominus$ is not closed on the set M. This has an influence on possible differences on dense subsets of [0, 1]. For example if $M = Q \cap [0, 1]$ and $(M, \ominus, \le, 0, 1)$ is a difference poset, then $\ominus$ is generated by a unique generator g; see Theorem 2. This generator should preserve the rationality, i.e., if $r, s \in Q$, $0 \le r \le s \le 1$, then

$$g^{-1}(g(s) - g(r)) \in Q$$

Example 2. Let

$$g_\lambda(x) = \frac{x}{\lambda x + (1 - \lambda)}, \qquad x \in [0, 1], \quad \lambda \in [0, 1)$$

Then the function g_λ is a rationality-preserving norm generator.

Open Problem. Are there some other rationality-preserving normed generators which are not included in the family $(g_\lambda)_{\lambda \in [0, 1)}$?

All previous results (Theorems 1 and 2), also can be extended for RI-posets of Kalmbach and Riečanova (n.d.). Note that an RI-poset $(X, \ominus, \leq, \mathbf{0})$ fulfills the same axioms as a D-poset up to the presence of a maximal element $\mathbf{1}$ in X. By Mesiar (n.d.-*b*); $([0, 1), \ominus, \leq, 0)$ is an RI-poset if and only if there is a generator g, g: $[0, 1) \to [0, \infty)$, $g(0) = 0$, and g is a continuous, strictly increasing function such that for $0 \leq a \leq b < 1$ the equality

$$b \ominus a = g^{-1}(g(b) - g(a))$$

holds.

Note that g is unique up to a positive multiplicative constant. If g is bounded, then the underlying RI-poset (on $[0, 1]$) can be (uniquely!) extended to a D-poset (on $[0, 1)$). If g is unbounded, then this extension is impossible. Theorem 2 holds true for any RI-poset on a dense subset M of the interval $[0, 1)$.

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